# An Hilbertian Framework for the Time-Continuous Monge-Kantorovich Problem

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*(Résumé : tsvp)*

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# Un cadre hilbertien pour le problème de Monge-Kantorovich continu en temps.

**Résumé :** Dans [1], une reformulation en temps continu a été introduite pour résoudre le problème de Monge-Kantorovich. Mais bien que la méthode décrite, basée sur un Lagrangien augmenté, suppose un cadre de travail Hilbertien, les calculs présentés étaient purement formels. Le problème Hilbertien manque de coercivité, et il est nécessaire d'utiliser les résultats récents effectués dans le domaine du transport de masse pour montrer l'existence d'un minimiseur. Après cette première étape, et en supposant que la densité optimale reste strictement positive, nous prouvons l'existence de points-selles pour les deux Lagrangiens définis en [1], avant de montrer la convergence de la méthode numérique.

In [1], a computational fluid dynamic style reformulation is introduced to solve the classical Monge-Kantorovich problem. Though the described augmented Lagrangian method involves an Hilbertian framework, the discussion was purely formal. Taking advantage of the recent progress in the optimal transport theory [4],[5],[6],[10] and despite the lack of coercivity of the Hilbertian problem, we establish an existence result. Then under a reasonable assumption of positivity for the density, we prove the existence of saddle-points for both Lagrangian defined in [1], and finally prove the convergence of the numerical method.

# Introduction

Given two non-negative density functions $\rho_0$ and $\rho_T$ on $\mathbb{R}^d$ satisfying the compatibility condition

$$\int_{\mathbb{R}^d} \rho_0(x)dx = \int_{\mathbb{R}^d} \rho_T(x)dx \ , \tag{1}$$

a map $M$ is said to transport $\rho_0$ to $\rho_T$ if, for any bounded subset $A$ of $\mathbb{R}^d$,

$$\int_A \rho_T(x)dx = \int_{M(A)} \rho_0(x)dx \ . \tag{2}$$

Then the Monge-Kantorovich problem (MKP) consists in finding a map $M$ transporting $\rho_0$ to $\rho_T$ and minimizing the cost

$$\int_{\mathbb{R}^d} |M(x) - x|^2 \rho_0(x)dx \ . \tag{3}$$

The problem of the existence and the characterization of the optimal map has been solved in [2] by Y. Brenier, who showed that the optimal map is the gradient of a convex potential. This result has been then extended in the case of more general cost functions in [8] and in the case of more general geometries in [10]. From a numerical point of view, the computation of the optimal map seems to be a challenging problem (see [1] for a brief review of existing methods). In their work [1], J.D. Benamou and Y. Brenier used an artificial (time) variable to linearize the constraints (2). Then an augmented numerical resolution of the resulting problem was presented. Although the optimal mass transport problem is naturally set up in the frame of probability measures and continuous test functions, the augmented method used in [1] is largely of Hilbertian nature. In order to prove the convergence of the method, it is therefore natural to study the optimal mass transport problem from an Hilbertian point of view. To achieve this goal, we use the regularity theory developed for the optimal mass transport problem by L.A. Caffarelli in the case of convex bounded domains and recently extended by D. Cordero-Erausquin in the case of the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. As in [1], we will consider only the case of the torus, which simplifies considerably the analysis. Now we define the time-continuous Monge-Kantorovich problem :

Let $\mathbb{T}^d$ be the $d$-dimensional unit cube with periodic boundary and define $Q = [0;T] \times \mathbb{T}^d$. In this study, we note

$$H(Q;div) = \{f \in L^2(Q)^{1+d} \ s.t. \ \nabla_{t,x}.f \in L^2(Q)\} \ , \tag{4}$$

$$V(Q) = \{f \in L^2(Q)^{1+d} \ s.t. \ \|\nabla_{t,x}.f\|_{L^2(Q)} = 0\} \ . \tag{5}$$

Given two densities $(\rho_0, \rho_T)$ in $L^2(\mathbb{T}^d)$ satisfying the compatibility condition (1), the time-continuous Monge-Kantorovich problem (TCMKP) is to minimize

$$K(\rho, m) = \int_0^T \int_{\mathbb{T}^d} \frac{|m|^2}{2\rho} dxdt \ , \tag{6}$$

over all pairs $(\rho, m)$ in $V(Q)$ satisfying the boundary conditions

$$\begin{aligned} \rho(0,.) &= \rho_0 \ in \ L^2(\mathbb{T}^d) \ , \\ \rho(T,.) &= \rho_T \ in \ L^2(\mathbb{T}^d) \ . \end{aligned} \tag{7}$$

We denote by $E(\rho_0, \rho_T)$ this minimum.

**Remark 0.1.** *The link between the (MKP) and the (TCMKP) may be unclear. It is used as a important tool in the proof of the main theorem of section 1, for which we refer for some more explanations.*

In this paper, our aim is to derive a rigorous Hilbertian theory for the (TCMKP), and to prove the convergence of the augmented Lagrangian method used in [1]. Then section 1 deals with the well posedness of the Hilbertian problem. Under fairly general hypotheses on the data, we show the existence of a minimizer of the (TCMKP), and link the optimal cost $E(\rho_0, \rho_T)$ to the Wasserstein distance between $\rho_0$ and $\rho_T$. In section 2, we look at the Lagrangian formulation of the (TCMKP) and prove an abstract existence result of a saddle-point. Since this Lagrangian $(L)$ is the starting point of the numerical method, we expect this result to be an important step when looking for a convergence result. Then the purpose of section 3 is to get more informations on the saddle-point. Precisely, we show that the Lagrange multiplier of the mass conservation constraint is linked to the optimal pair $(\rho^*, m^*)$. Then we remind the second Lagrangian $(\mathcal{L})$ introduced in [1], and use the saddle-points of $L$ to characterize the saddle-points of $\mathcal{L}$. We get therefore an existence result for a saddle-point of $\mathcal{L}$. Finally, section 4 presents a convergence result for the numerical algorithm. This result may be unexpected since some of the classical assumptions required for convergence are not fulfilled.

# 1 Well posedness of the Hilbertian problem

In this section, our main result is the existence of a minimizer of the (TCMKP) and the characterization of the optimal cost $E(\rho_0, \rho_T)$. Those results are summarized in the following theorem.

**Theorem 1.1.** *Assume that the time-boundary data $\rho_0$ and $\rho_T$ satisfy the following properties*

$$\begin{aligned} &\forall x \in \mathbb{T}^d, \ 0 < \alpha \le \rho_0(x) \le M \ , \\ &\forall x \in \mathbb{T}^d, \ 0 < \alpha \le \rho_T(x) \le M \ . \end{aligned} \tag{8}$$

*Then $E(\rho_0, \rho_T)$ satisfies*

$$E(\rho_0, \rho_T) = \frac{1}{2T} d^2_{Wass}(\rho_0, \rho_T) \ . \tag{9}$$

*Moreover, there exists a minimizer $(\rho^*, m^*)$ satisfying*

$$\|\rho^*\|_{L^\infty([0;T];L^\infty(\mathbb{T}^d))} \leq M \ , \tag{10}$$

$$\|m^*\|_{L^\infty(]0;T[;L^\infty(\mathbb{T}^d))} \leq M \frac{\sqrt{d}}{T} \ . \tag{11}$$

**Sketch of the proof :** first, we give a precise definition for the kinetic energy $K(\rho, m)$ defined in (6). The idea of the proof is then to consider a minimizing sequence $(\rho_n, m_n)$ and to use it to build a bounded minimizing sequence $(\tilde{\rho_n}, \tilde{m_n})$. This allows the extraction of a converging subsequence, which leads to an effective minimizer of the (TCMKP). This construction is done in five steps.

- Step 1 : Using lemma 1.3, we define a reference density $\bar{\rho}$ which is both bounded from above and from below away from zero.
- Step 2 : Let $(\rho_n, m_n)$ be a minimizing sequence. An appropriated convex combination of $\rho_n$ and $\bar{\rho}$ is used to build a new minimizing sequence $(\rho_n^{(1)}, m_n^{(1)})$, whose densities are bounded from below away from zero for each $n$.
- Step 3 : A new sequence $(\rho_n^{(2)}, m_n^{(2)})$ is built through a careful regularization process : it is of primary important to keep the time-boundary data of $\rho_n^{(2)}$ very close to the original ones.
- Step 4 : The regularity of $\rho_n^{(2)}$ and $m_n^{(2)}$ and the positivity of $\rho_n^{(2)}$ allow to define a smooth velocity field $v_n^{(2)}$. Then using the characteristics, we reduce the (TCMKP) to the (MKP), and use the regularity theory for this problem. Indeed, when the time-boundary data are smooth, the optimal map can be used to define an interpolated density which is optimal. An argument of R.J. McCann (see lemma 1.5) allows then to bound a new sequence $(\rho_n^{(3)}, m_n^{(3)})$.
- Step 5 : $(\rho_n^{(3)}, m_n^{(3)})$, just like $(\rho_n^{(2)}, m_n^{(2)})$, does not satisfy the original time-boundary conditions (7). This difficulty can be avoided using some "boundary-layers". Finally, the minimizing sequence $(\tilde{\rho_n}, \tilde{m_n})$ is obtained.

### A rigorous definition of the kinetic energy

This definition follows from the observation that for positive $\rho$

$$\frac{|m|^2}{2\rho} = \sup_{a + \frac{|b|^2}{2} \leq 0} [a\rho + b.m] \ . \tag{12}$$

Then $K(\rho, m)$ can be defined trough the following equality

$$K(\rho, m) = \sup \int_Q a\rho + b.m \ , \tag{13}$$

where $(a, b) \in L^2(Q) \times L^2(Q)^d$ are subject to the constraint that for all non-negative $f \in L^\infty(Q)$

$$\int_0^T \int_{\mathbb{T}^d} f(a + \frac{|b|^2}{2}) dx \leq 0 \ . \tag{14}$$

We denote by $\tilde{K}$ the set of all pairs $(a, b) \in L^2(Q)^{1+d}$ satisfying (14). It is easy to see that $\tilde{K}$ is a closed convex set in $L^2(Q)^{1+d}$. Moreover, for any fixed pair $(a, b) \in \tilde{K}$, the application $(\rho, m) \mapsto \int_Q a\rho + b.m$ is convex and continuous, and then lower semi continuous. Then the higher envelope $K$ of those functions is still convex and l.s.c. This property of $K$ will prove to be useful in the sequel.

**Remark 1.2.** *Such a characterization of the kinetic energy has been used by Brenier in [3]. The test functions were taken in $\mathcal{C}^0(Q)$, so that the interior of $\tilde{K}$ was not empty. This property allowed Brenier to use a duality theorem of Rockafellar to get the existence of a minimizer. However, this minimizer was found in the set of Radon measures. Since we expect more regularity for our minimizer, our definition of $\tilde{K}$ is slightly different, and this set turns to be of empty interior.*

**First step : definition of the reference density**

In order to define this density, we first state the following useful lemma.

**Lemma 1.3.** *Let $(\rho_0, \rho_T)$ be two densities in $L^2(\mathbb{T}^d)$ such that (1) holds. Assume that there exists $\alpha$ in $\mathbb{R}$ such that :*

$$\begin{aligned} &\forall x \in \mathbb{T}^d, \ 0 < \alpha \leq \rho_0(x) \ , \\ &\forall x \in \mathbb{T}^d, \ 0 < \alpha \leq \rho_T(x) \ . \end{aligned} \tag{15}$$

*Then there exists $(\bar{\rho}, \bar{m})$ in $V(Q)$ such that*

$$K(\bar{\rho}, \bar{m}) \leq \frac{C}{\alpha T^2} \|\rho_0 - \rho_n\|^2_{L^2(\mathbb{T}^d)} \ . \tag{16}$$

*Proof.* Define $\bar{\rho}(t, .) = \dfrac{(T-t)}{T}\rho_0(.) + \dfrac{t}{T}\rho_T(.)$.

Let $L^2_0(\mathbb{T}^d) = \{f \in L^2(\mathbb{T}^d) \ s.t. \ \int_{\mathbb{T}^d} f(x)dx = 0\}$. We consider the solution $\psi$ of the elliptic problem

$$\begin{cases} \Delta_x \psi = \dfrac{1}{T}(\rho_0 - \rho_T) \in \mathbb{T}^d \ , \\ \psi \in H^1(\mathbb{T}^d) \cap L^2_0(\mathbb{T}^d) \ . \end{cases}$$

Define then $\bar{m}(t,.) = \overrightarrow{\nabla}_x \psi$. We have $\|\bar{m}\|_{L^2(Q)} < \infty$ and by construction, $(\bar{\rho}, \bar{m})$ is in $V(Q)$. Finally, noticing that for all $(t,x)$ in $Q$, we have $\alpha \leq \bar{\rho}(t,x)$, we get

$$K(\bar{\rho}, \bar{m}) \leq \frac{1}{\alpha} \|\bar{m}\|^2_{L^2(Q)} \leq \frac{C}{\alpha T^2} \|\rho_0 - \rho_T\|^2_{L^2(Q)} \ . \tag{17}$$

This achieves the proof of the lemma. □

**Remark 1.4.** *(17) still holds with* $\|\rho_0 - \rho_T\|^2_{H^{-1}(Q)}$ *instead of* $\|\rho_0 - \rho_T\|^2_{L^2(Q)}$. *Moreover, we see that if* $\rho_0$ *and* $\rho_T$ *are in* $L^\infty(\mathbb{T}^d)$*, then*

$$\|\bar{\rho}\|_{L^\infty(Q)} \leq \max\left(\|\rho_0\|_{L^\infty(\mathbb{T}^d)}, \|\rho_T\|_{L^\infty(\mathbb{T}^d)}\right) \ .$$

**Second step : construction of** $(\rho_n^{(1)}, m_n^{(1)})$

Let $(\rho_n, m_n)_{n\in\mathbb{N}}$ be a minimizing sequence for the (TCMKP). Let $(\bar{\rho}, \bar{m})$ be the pair defined in the lemma. For any $n$ in $\mathbb{N}^*$, we define

$$\left\{\begin{array}{l} \rho_n^{(1)} = \dfrac{n-1}{n}\rho_n + \dfrac{1}{n}\bar{\rho} \ , \\[2ex] m_n^{(1)} = \dfrac{n-1}{n} m_n + \dfrac{1}{n}\bar{m} \ . \end{array}\right. \tag{18}$$

By construction, we have $(\rho_n^{(1)}, m_n^{(1)}) \in V(Q)$ and

$$\forall (t,x) \in Q, \rho_n^{(1)}(t,x) \geq \frac{\alpha}{n} \ . \tag{19}$$

The convexity of $K$ gives

$$K(\rho_n^{(1)}, m_n^{(1)}) \leq \frac{n-1}{n} K(\rho_n, m_n) + \frac{1}{n} K(\bar{\rho}, \bar{m}) \ , \tag{20}$$

so that $(\rho_n^{(1)}, m_n^{(1)})$ is still a minimizing sequence.

**Third step : construction of** $(\rho_n^{(2)}, m_n^{(2)})$

Now we want to build a smooth "outer" (in the sense that the time-boundary data are not satisfied, but only carefully approximated) minimizing sequence. This would be necessary to define a smooth velocity field and some characteristics. Those characteristics would then be used as an essential tool to bound an appropriated minimizing sequence.

Define $f$ in $\mathcal{C}(\mathbb{R} \times \mathbb{T}^d)$ as follows

$$\left\{\begin{array}{lll} f(t,x) = (0,x) & if & t < 0 \ , \\[2ex] f(t,x) = (t,x) & if & t \in [0;T] \ , \\[2ex] f(t,x) = (T,x) & if & t > T \ . \end{array}\right. \tag{21}$$

Define $\rho_n^* = \rho_n^{(1)} \circ f$ and $m_n^* = \chi_{[0;T]}[m_n^{(1)} \circ f]$. We have $(\rho_n^*, m_n^*) \in V(Q)$. This extension of the functions allows a good convergence of the time-boundary values.

Fix $n \in \mathbb{N}^*$. For any $k$ in $\mathbb{N}^*$, we define

$$g_k(t,x) = (-\frac{T}{k} + (1+\frac{2}{k})t, x)\ . \tag{22}$$

The sequence $(\rho_n^k, m_n^k)$ is then defined as follows

$$\begin{cases} \rho_n^k(t,x) = \xi_{\frac{T}{k}} * [\rho_n^* \circ g_k]\ , \\ \\ m_n^k(t,x) = (1+\frac{2}{k})\ \xi_{\frac{T}{k}} * [m_n^* \circ g_k]\ , \end{cases} \tag{23}$$

where $\xi$ is a positive mollifier with support contained in the unit ball of $\mathbb{R}^{d+1}$.
It is easy to see that for any $k$ in $\mathbb{N}^*$, $(\rho_n^k, m_n^k)$ is in $V(Q)$. Moreover, the pair $(\rho_n^k, m_n^k)$ satisfies the following properties

$$\begin{cases} \rho_n^k \in \mathcal{C}^\infty(Q)\ , \\ \\ m_n^k \in \mathcal{C}^\infty(Q)^d\ , \\ \\ \forall (t,x) \in Q,\ \dfrac{\alpha}{n} \leq \rho_n^k(t,x)\ , \end{cases} \tag{24}$$

and some sub-sequence (still labeled by $k$) satisfies

$$\begin{cases} \lim_{k\to\infty} \|(\rho_n^k, m_n^k) - (\rho_n^{(1)}, m^{(1)})\|_{H(Q;div)} = 0\ , \\ \\ \lim_{k\to\infty} \|\rho_n^k(0,.) - \rho_0\|_{L^2(\mathbb{T}^d)} = 0\ , \\ \\ \lim_{k\to\infty} \|\rho_n^k(T,.) - \rho_T\|_{L^2(\mathbb{T}^d)} = 0\ . \end{cases} \tag{25}$$

The convergence of $K(\rho_n^k, m_n^k)$ towards $K(\rho_n^{(1)}, m_n^{(1)})$ follows simply from the lower semi continuity of $K$. We get then that there exists a $k_n$ in $\mathbb{N}$ such that

$$K(\rho_n^{k_n}, m_n^{k_n}) \leq K(\rho_n^{(1)}, m_n^{(1)}) + \frac{1}{n}\ , \tag{26}$$

$$\|\rho_n^{k_n}(0,.) - \rho_0\|_{L^2(\mathbb{T}^d)} \leq \frac{1}{n^2}\ , \tag{27}$$

$$\|\rho_n^{k_n}(T,.) - \rho_T\|_{L^2(\mathbb{T}^d)} \leq \frac{1}{n^2}\ . \tag{28}$$

Define $(\rho_n^{(2)}, m_n^{(2)}) = (\rho_n^{k_n}, m_n^{k_n})$. By construction, we have

$$\begin{cases} (\rho_n^{(2)}, m_n^{(2)}) \in \mathcal{C}^\infty(Q)^{d+1} \ , \\ \forall (t,x) \in Q, \ \rho_n^{(2)} \geq \dfrac{\alpha}{n} \ . \end{cases} \tag{29}$$

Moreover, (26) implies that

$$K(\rho_n^{(2)}, m_n^{(2)}) \leq K(\rho_n^{(1)}, m_n^{(1)}) + \frac{1}{n} \ . \tag{30}$$

Hence $(\rho_n^{(2)}, m_n^{(2)})$ is an "outer" minimizing sequence in the sense that the time-boundary conditions are only approximated.

**Fourth step : construction of** $(\rho_n^{(3)}, m_n^{(3)})$

In this section, we take advantage of the regularity of $\rho_n^{(3)}$ on the time-boundary. This allows to recover the (MKP) and to then to bound a new "outer" minimizing sequence. We therefore define now a velocity field

$$v_n^{(2)} = \frac{m_n^{(2)}}{\rho_n^{(2)}} \ . \tag{31}$$

$v_n^{(2)}$ is in $\mathcal{C}^\infty(Q)^d$. We are now able to define the characteristics. We look at the differential system

$$\begin{cases} \partial_t X(t,x) = v_n^{(2)}(t, X(t,x)) \ , \\ X(0,x) = x \ . \end{cases} \tag{32}$$

Since $v_n^{(2)}$ is a $\mathcal{C}^\infty$ function, this system is well defined and the solution is uniquely defined on $[0;T]$.

Now we remind the computations in [1] to show that the "optimal displacement" between $X(0,.)$ and $X(T,.)$ follows straight lines. Indeed, we have

$$\begin{aligned}
T\int_{\mathbb{T}^d}\int_0^T \rho_n^{(2)}(t,x)|v_n^{(2)}(t,x)|^2 dxdt &= T\int_{\mathbb{T}^d}\int_0^T \rho_n^{(2)}(0,x)|v_n^{(2)}(t,X(t,x))|^2 dxdt\ ,\\
&= T\int_{\mathbb{T}^d}\int_0^T \rho_n^{(2)}(0,x)|\partial_t X(t,x)|^2 dxdt\ ,\\
&\geq \int_{\mathbb{T}^d} \rho_n^{(2)}(0,x)|X(T,x)-X(0,x)|^2 dx\ ,\\
&= \int_{\mathbb{T}^d} \rho_n^{(2)}(0,x)|X(T,x)-x|^2 dx\ .
\end{aligned}$$

It is then sufficient to consider the Monge-Kantorovich problem between the densities $\rho_n^{(2)}(0,.)$ and $\rho_n^{(2)}(T,.)$. From [6], we get the existence of a convex function $\phi_n$ such that $\nabla_x\phi_n$ minimizes

$$\int_{\mathbb{T}^d} \rho_n^{(2)}(0,x)|M(x)-x|^2 dx \tag{33}$$

in the set of application $M$ pushing $\rho_n^{(2)}(0,.)$ forward to $\rho_n^{(2)}(T,.)$. This minimum is the definition of the Wasserstein distance between $\rho_n^{(2)}(0,.)$ and $\rho_n^{(2)}(T,.)$. We get then that

$$d^2_{Wass}(\rho_n^{(2)}(0,.),\rho_n^{(2)}(T,.)) \leq 2T\ K(\rho_n^{(2)},m_n^{(2)})\ . \tag{34}$$

More precisely, $\phi_n$ is convex in $\mathbb{R}^d$, and is additive in the sense that for any $p\in\mathbb{Z}^d$ and for almost any $x\in\mathbb{R}^d$

$$\nabla_x\phi_n(x+p) = \nabla_x\phi_n(x) + p\ . \tag{35}$$

The boundary data of this allocation problem are $\mathcal{C}^\infty$, what allows us to use the regularity theory developed in [4],[5], and extended in the case of the torus in [6]. We have that the function $\phi_n$ is in $\mathcal{C}^{2,\beta}(\mathbb{T}^d)$ for some $0<\beta<1$. We can then define point-wise the "interpolated density" $\rho_n^{(3)}$ between $\rho_n^{(2)}(0,.)$ and $\rho_n^{(2)}(T,.)$. Then for every $(t,x)$ in $Q$ we note

$$\rho_n^{(3)}\left(t,\frac{T-t}{T}x+\frac{t}{T}\nabla_x\phi_n(x)\right)\det\left(\frac{T-t}{T}I_d+\frac{t}{T}D^2\phi_n(x)\right) = \rho_n^{(2)}(0,x)\ . \tag{36}$$

This equality implies that $\phi_n$ is strictly convex in $\mathbb{T}^d$. For any density defined in the same way as in (36), R.J. McCann proved in [9] that for $1\leq p\leq\infty$, the $L^p$-norms are dominated by the $L^p$-norms of the data. We prove here the result in the $L^\infty$-case.

**Lemma 1.5.** *Let $S$ be a positive definite symmetric matrix of size $d$.*
*Let $S(t) = (1-t)Id + tS$ for any $t$ in $[0;1]$. Define then $v(t) = det(S_t)$. Then for every $t$ in $[0;1]$, we have*

$$v(t) \geq \det(Id)^{1-t} \det(S)^t \ . \tag{37}$$

*Proof.* The proof of the lemma simply follows from a convexity inequality. Indeed, if $(\lambda_i)_{i=1..d}$ are the eigenvalues of $S$, then $((1-t)+t\lambda_i)_{i=1..d}$ are the eigenvalues of $S_t$. For every $i$ in $[0;d]$, we have

$$(1-t) + t\lambda_i \geq 1^{1-t}\lambda_i^t \ . \tag{38}$$

Then $\det(S(t)) \geq \det(S)^t$ and the lemma is proved. □

In our case, we have that $\det D^2\phi_n(x) = \dfrac{\rho_n^{(2)}(0,x)}{\rho_n^{(2)}(T, \nabla_x\phi_n(x))}$. We deduce that

$$\forall (t,x) \in Q, \ \rho_n^{(3)}(M_t(x)) \leq \rho_n^{(2)}(0,x)^{\frac{T-t}{T}} \rho_n^{(2)}(T,x)^{\frac{t}{T}} \ , \tag{39}$$

where $M_t$ is the application $x \mapsto \frac{T-t}{T}x + \frac{t}{T}\nabla_x\phi_n(x)$.

We have then that $\|\rho_n^{(3)}\|_{L^\infty([0;T];L^\infty(D))} \leq M$. We get that the densities $(\rho_n^{(3)})_{n\in\mathbb{N}^*}$ are uniformly bounded in $L^\infty(Q)$.

We define now the velocity associated to this particular map. We have

$$\forall (t,x) \in Q, \ v_n^{(3)}(t, M_t(x)) = \frac{1}{T}(\nabla_x\phi_n(x) - x) \ . \tag{40}$$

Since $\nabla_x\phi_n$ minimizes (33), we have $\|\nabla_x\phi_n(x) - x\| \leq \sqrt{d}$ for any $x$ in $\mathbb{T}^d$. We have therefore that $\|v_n^{(3)}\|_{L^\infty([0;T];L^\infty(D))} \leq \frac{\sqrt{d}}{T}$. And then

$$\|m_n^{(3)}\|_{L^\infty([0;T];L^\infty(Q))} \leq M\frac{\sqrt{d}}{T} \ . \tag{41}$$

It is easy to see that the mapping $M_t$ derives from a strictly convex potential satisfying the additive property (35). Hence we get that $M_t(\mathbb{T}^d) = \mathbb{T}^d$, so that equation (40) defines $v_n^{(3)}$ everywhere in $Q$. Moreover, $v_n^{(3)}$ is of class $\mathcal{C}^\infty$ and the function $m_n^{(3)}$ defined by $m_n^{(3)} = \rho_n^{(3)}v_n^{(3)}$ satisfies $\partial_t\rho_n^{(3)} + \nabla_x.m_n^{(3)} = 0$ in the classical sense in $Q$. Then using the same calculation as for $(\rho_n^{(2)}, v_n^{(2)})$, we get

$$d^2_{Wass}(\rho_n^{(2)}(0,.), \rho_n^{(2)}(T,.)) = 2T \ K(\rho_n^{(3)}, m_n^{(3)}) \ . \tag{42}$$

Summarizing (20),(26),(34),(42), we get

$$\begin{aligned}
\frac{1}{2T}d^2_{Wass}(\rho_n^{(2)}(0,.),\rho_n^{(2)}(T,.)) &= K(\rho_n^{(3)},m_n^{(3)}) \\
&\leq K(\rho_n^{(2)},m_n^{(2)}) \ , \\
&\leq K(\rho_n^{(1)},m_n^{(1)}) + \frac{1}{n} \ , \\
&\leq K(\rho_n,m_n) + \frac{K(\bar\rho,\bar m)+1}{n} \ .
\end{aligned}$$

The Wasserstein distance is continuous with respect to the $L^2$ distance, so we have that $d^2_{Wass}(\rho_n^{(2)}(0,.),\rho_n^{(2)}(T,.))$ converges to $d^2_{Wass}(\rho_0,\rho_T)$ as $n$ goes to infinity. Passing to the limit in the previous inequality, we get

$$\frac{1}{2T}d^2_{Wass}(\rho_0,\rho_T) \leq E(\rho_0,\rho_T) \ . \tag{43}$$

As we will see, it is possible to get the converse inequality. We just have to build a new "admissible" sequence (since $\rho_n^{(3)}(0,.)$ and $\rho_n^{(3)}(0,.)$ do not satisfy the time-boundary conditions (7)).

**Fifth step : some "boundary layers" and the final sequence**

Let $\delta > 0$. We define the function $\tilde\rho_n$ as follows

$$\begin{aligned}
\tilde\rho_n(t,.) &= \tfrac{(\delta T - t)}{\delta T}\rho_0(.) + \tfrac{t}{\delta T}\rho_n^{(2)}(0,.) && \text{for} \quad 0 \leq t \leq \delta T \ , \\
\tilde\rho_n(t,.) &= \rho_n^{(3)}(\tfrac{t-\delta T}{T-2\delta T},.) && \text{for} \quad \delta T \leq t \leq T-\delta T \ , \\
\tilde\rho_n(t,.) &= \tfrac{(T-t)}{\delta T}\rho_n^{(2)}(T,.) + \tfrac{t-T+\delta T}{\delta T}\rho_T(.) && \text{for} \quad T-\delta T \leq t \leq T \ .
\end{aligned} \tag{44}$$

The function $\tilde m_n$ is defined on the time-intervals $[0;\delta T]$ and $[(1-\delta)T;T]$ as in lemma 1.3 and in $[\delta T;(1-\delta)T]$ by a re-normalization and a time rescaling in a very similar way as (23).

By construction, the pair $(\tilde{\rho}_n, \tilde{m}_n)$ is in $V(Q)$ and satisfies the boundary conditions (7). Moreover, using (16),(27),(28),(42), we get

$$\begin{aligned} K(\tilde{\rho}_n, \tilde{m}_n) &\leq \frac{C}{(\delta T)^2}\frac{n}{\alpha}(\|\rho_0 - \rho_n^{(2)}(0)\|^2_{L^2(\mathbb{T}^d)} + \|\rho_T - \rho_n^{(2)}(T)\|^2_{L^2(\mathbb{T}^d)}) \\ &\quad + \frac{1}{1-2\delta}K(\rho_n^{(3)}, m_n^{(3)}) \ , \qquad (45) \\ &\leq \frac{2C}{n^3\alpha(\delta T)^2} + \frac{1}{1-2\delta}\frac{1}{2T}d^2_{Wass}(\rho_0, \rho_T) \ . \end{aligned}$$

We see that $\delta = n^{-1}$ is a good choice, and leads to

$$K(\tilde{\rho}_n, \tilde{m}_n) \leq \frac{2C}{n\alpha T^2} + \frac{n}{n-2}\frac{1}{2T}d^2_{Wass}(\rho_0, \rho_T) \ . \tag{46}$$

Passing to the limit in the previous inequality, and using that since the sequence $(\tilde{\rho}_n, \tilde{m}_n)_{n\in\mathbb{N}^*}$ is admissible, $K(\tilde{\rho}_n, \tilde{m}_n) \geq E(\rho_0, \rho_T)$, we see that

$$E(\rho_0, \rho_T) = \frac{1}{2T}d^2_{Wass}(\rho_0, \rho_T) \ . \tag{47}$$

From our construction, we see that the

$$\|\tilde{\rho}_n\|_{L^\infty([0;T];L^\infty(Q))} \leq M \ . \tag{48}$$

Moreover, the sequence $(K(\tilde{\rho}_n, \tilde{m}_n))_{n\in\mathbb{N}^*}$ is bounded. Then we have

$$\|\tilde{m}_n\|_{L^2(Q)} \leq \sqrt{\|\tilde{\rho}_n\|_{L^\infty(Q)}}\sqrt{2K(\tilde{\rho}_n, \tilde{m}_n)} \ . \tag{49}$$

We deduce that $(\tilde{m}_n)_{n\in\mathbb{N}^*}$ is bounded in $L^2(Q)$. We can therefore extract a weakly converging subsequence in $H(Q, div)$. We denote the limit by $(\rho^*, m^*)$. We have $(\rho^*, m^*) \in V(Q)$. From the lower semi continuity of $K$, we get

$$K(\rho^*, m^*) \leq E(\rho_0, \rho_T) \ . \tag{50}$$

Then by definition of $E(\rho_0, \rho_T)$ and the fact that $(\rho^*, m^*)$ is admissible, we have an equality in the previous inequality. Thus the Hilbertian time-continuous Monge-Kantorovich problem admits a minimizer. Moreover, this minimizer satisfies the following estimates

$$\|\rho^*\|_{L^\infty([0;T];L^\infty(\mathbb{T}^d))} \leq M \ , \tag{51}$$

$$\|m^*\|_{L^\infty(]0;T[;L^\infty(\mathbb{T}^d))} \leq M\frac{\sqrt{d}}{T} \ . \tag{52}$$

This achieves the proof of theorem 1.1.

## 2 Existence of a saddle-point for the Lagrangian

From now on, we will assume that the optimal density is bounded from below away from 0. Notice that since the time-boundary densities satisfy (8), it is always satisfied for smooth data. Indeed, in this particular case, the optimal density is simply deduced from the optimal map by equation (36). Then since $\phi$ is in $\mathcal{C}^2(\mathbb{T}^d)$ (see [6]) and has a strictly positive Hessian, any of the eigenvalues is strictly positive. Those eigenvalues vary continuously in $\mathbb{T}^d$. Since $\mathbb{T}^d$ is compact, we conclude that any eigenvalue is bounded away from zero. We now state the main result of this section.

**Theorem 2.1.** *Assume that the solution $(\rho^*, m^*)$ of the (TCMKP) satisfies*

$$\rho^*(x,t) \geq \alpha_1, \ \forall (t,x) \in Q \ , \tag{53}$$

*for some $\alpha_1 > 0$.*
*Then there exists a $\lambda^* \in L_0^2(Q)$ such that $(\rho^*, m^*, \lambda^*)$ is a saddle-point of the Lagrangian $L$, defined as follows*

$$L(\rho, m, \lambda) = K(\rho, m) + \int_0^T \int_D (\partial_t \rho + \nabla_x . m) \lambda \ dxdt \ . \tag{54}$$

The proof of this theorem relies mainly on an application of the Hahn-Banach theorem. The two convex sets we want to separate are defined as follows

$$S = \{(K(\rho,m) - K(\rho^*, m^*) + s, \partial_t \rho + \nabla_x . m), (\rho, m) \in H(Q; div), s \geq 0\} \ ,$$

$$T = \left\{(-t, 0) \in \mathbb{R} \times L_0^2(Q), t > 0\right\} \ .$$

The next three lemmas show that $S$ and $T$ satisfy the required assumptions for the Hahn-Banach theorem.

**Lemma 2.2.** *$S$ and $T$ are convex.*

*Proof.* The convexity of $T$ is obvious. Let $(r_1, \psi_1)$ and $(r_2, \psi_2)$ be in S. There exists $(\rho_1, m_1, s_1)$ and $(\rho_2, m_2, s_2)$ such that

$$(K(\rho_i, m_i) - K(\rho^*, m^*) + s_i, \partial_t \rho + \nabla_x . m) = (r_i, \psi_i) \ . \tag{55}$$

From the convexity of $K$, we get

$$K(\frac{1}{2}(\rho_1 + \rho_2, m_1 + m_2)) \leq \frac{1}{2}(K(\rho_1, m_1) + K(\rho_2, m_2)) \ . \tag{56}$$

Let $s_3 = \frac{1}{2}(s_1 + s_2) + \frac{1}{2}(K(\rho_1, m_1) + K(\rho_2, m_2)) - K(\frac{1}{2}(\rho_1 + \rho_2, m_1 + m_2))$.

We have $s_3 \geq 0$. Define then

$$(\rho_3, m_3) = \tfrac{1}{2}(\rho_1 + \rho_2, m_1 + m_2) ,$$

$$\psi_3 = \tfrac{1}{2}(\psi_1 + \psi_2) ,$$

$$r_3 = \tfrac{1}{2}(r_1 + r_2) .$$

From the linearity of the divergence operator, we get $\psi_3 = \nabla_{t,x}.(\rho_3, m_3)$. Then we have that $s_3 \geq 0$ and $\psi_3 \in H(Q; div)$ such that

$$(K(\rho_3, m_3) - K(\rho^*, m^*) + s_3, \partial_t \rho + \nabla_x .m) = (r_3, \psi_3) .$$

Hence $(r_3, \psi_3)$ is in $S$. This proves the convexity of $S$. □

**Lemma 2.3.** $S \cap T = \emptyset$.

*Proof.* Let $(r, \psi)$ be in $S \cap T$. We have $\psi = 0$. Let $(\rho, m, s)$ such that

$$(K(\rho, m) - K(\rho^*, m^*) + s, \partial_t \rho + \nabla_x .m) = (r, 0) . \tag{57}$$

Using the definition of $(\rho^*, m^*)$, we have $r \geq s$. Hence $r \geq 0$. But we should have $r < 0$ since $(r, \psi)$ is in $T$. This is a contradiction. We conclude that $S \cap T = \emptyset$. □

**Lemma 2.4.** *The interior of $S$ is not empty.*

*Proof.* Let $s_0 > 0$. As we will show, $(s_0, 0)$ is an interior point of $S$.

Let $0 < \epsilon < 1/2$. Take $(r, g)$ in a neighborhood of $(s_0, 0)$, that is such that

$$|r - s_0| + \|g\|_{L^2} < \epsilon . \tag{58}$$

We want to prove that $(r, g)$ is in $S$ for $\epsilon$ small enough. We are therefore looking for a $(\rho, m, s)$ such that

$$\begin{cases} K(\rho, m) - K(\rho^*, m^*) + s_1 = s_0 , \\ \nabla_{t,x}.(\rho, m) = g . \end{cases}$$

One of the difficulties comes from the fact that $\rho$ has to satisfy some positivity property. In order to control the $L^\infty$-norm of the new density, we integrate the mass production induced by $g$. We define then $h(t) = \int_0^t \int_D g(u, x) dx du$. Since $g$ is in $L_0^2(Q)$, we have $h(T) = 0$. This condition is necessary to allow the recovery of the boundary conditions for $\rho$. Our strategy is then to split $\rho$ in two parts. The first part has to stay close to the optimal density while the second has to track the mass production.

We then define $\rho_2(t,x) = \theta\alpha_1 + h(t)$, for some $\theta$. Then we must have the following equality

$$\nabla_x . m(t,x) = g(t,x) - \int_D g(t,y)dy \ . \tag{59}$$

For a.e. $t$ in $[0;T]$, we solve the system

$$\begin{cases} \Delta_x(\psi_t) = g(t,x) - \int_D g(t,y)dy \ , \\ \overrightarrow{\nabla}\psi_t . \overrightarrow{n} = 0 \ , \\ \psi_t \in H^1(D) \cap L_0^2(D) \ . \end{cases}$$

We take then $\tilde{m}(t,x) = \nabla_x \psi_t(x)$. By construction, we have

$$\|\tilde{m}(t,.)\|_{L^2(D)} \leq C\|g(t,.)\|_{L^2(D)} \ . \tag{60}$$

Integrating in $t$, we get $\|\tilde{m}\|_{L^2(Q)} \leq C\|g\|_{L^2(Q)}$ .

**Remark 2.5.** *This construction does not take care of the measurability of the resulting function $\tilde{m}$, which could be easily stated. Anyway, the bound (60) allows some regularization process...*

Now we consider $K(\rho^* + h(t), m^* + \tilde{m})$. We want to prove that this action is close to the minimum. Therefore, we define

$$(\rho_1, m_1) = (\rho^* - \theta\alpha_1, m^*) \ , \tag{61}$$

$$(\rho_2, m_2) = (h(t) + \theta\alpha_1, \tilde{m}) \ . \tag{62}$$

The inequality $K((\rho_1,m_1) + (\rho_2,m_2)) \leq K(\rho_1,m_1) + K(\rho_2,m_2)$ follows from the convexity and the homogeneity property of $K$.

$$\begin{aligned} K(\rho_1, m_1) &= \int_D \int_0^T \frac{|m^*|^2}{2(\rho^* - \theta\alpha_1)} dxdt \ , \\ &= \int_D \int_0^T \frac{|m^*|^2}{2\rho^*} \frac{\rho^*}{\rho^* - \theta\alpha_1} dxdt \ . \end{aligned}$$

But we have $\alpha_1 \leq \rho^*$, so that $\rho^* - \theta\alpha_1 \geq (1-\theta)\rho^*$. Hence we have

$$K(\rho_1, m_1) \leq \frac{1}{1-\theta} K(\rho^*, m^*) \ .$$

Finally, we get $K(\rho_1, m_1) - K(\rho^*, m^*) \leq \frac{\theta}{1-\theta} K(\rho^*, m^*)$.

Now we have to estimate $K(\rho_2, m_2)$. We have

$$\rho_2(t,x) = \int_0^t \int_D g(u,x)dxdu + \theta\alpha_1 \ , \tag{63}$$

$$= \int_0^T \int_D g(u,x)\chi_{[0;t]}dxdu + \theta\alpha_1 \ . \tag{64}$$

Then for all $(t,x) \in Q$

$$\theta\alpha_1 - \|g\|_{L^2(Q)}\sqrt{t} \leq \rho_2(t,x) \leq \theta\alpha_1 + \|g\|_{L^2(Q)}\sqrt{t} \ .$$

Hence for $\|g\|_{L^2(Q)} \leq \frac{1}{\sqrt{T}}\theta\alpha_1$ we have

$$\forall (t,x) \in Q, \ \ \rho_2(t,x) \geq 0 \ .$$

Moreover, taking $2\epsilon\sqrt{T} \leq \alpha_1\theta$, we get

$$\begin{aligned} K(\rho_2, m_2) &\leq \int_0^T \int_D \frac{1}{\theta\alpha_1}|m_2|^2 \ , \\ &\leq \frac{1}{\theta\alpha_1}\|m_2\|^2_{L^2(Q)} \ , \\ &\leq \frac{C}{\theta\alpha_1}\|g\|^2_{L^2(Q)} \ . \end{aligned}$$

Finally, we get

$$\begin{aligned} K(\rho,m) - K(\rho^*,m^*) &\leq \frac{\theta}{1-\theta}K(\rho^*,m^*) + \frac{C}{\theta\alpha_1}\|g\|^2_{L^2(Q)} \ , \\ &\leq \frac{\theta}{1-\theta}K(\rho^*,m^*) + \frac{C}{\theta\alpha_1}\epsilon^2 \ , \\ &\leq 4\epsilon\frac{\sqrt{T}}{\alpha_1}K(\rho^*,m^*) + \frac{C}{2\sqrt{T}}\epsilon \text{ when taking } \theta = 2\epsilon\frac{\sqrt{T}}{\alpha_1} \ . \end{aligned}$$

We want $K(\rho,m) - K(\rho^*,m^*) \leq r$. It is sufficient to take a small $\epsilon$, since $r$ is of the same order as $s_0$. It is then possible to take $s = r - (K(\rho,m) - K(\rho^*,m^*))$, and we we finally get $(\rho, m, s)$ such that

$$(K(\rho,m) - K(\rho^*,m^*) + s, \partial_t\rho + \nabla_x.m) = (r,g) \ . \tag{65}$$

We deduce that the interior of $S$ is not empty. □

It is now possible to end the proof of the theorem. From the Hahn-Banach theorem, there exists a non-zero linear form separating $S$ and $T$. We have then $(\alpha_0, \phi_0) \in \mathbb{R} \times L_0^2$ such that

$$\forall(\rho, m, s, t),\ \alpha_0(K(\rho, m) - K(\rho^*, m^*) + s) + < \nabla_{t,x}.(\rho, m), \phi_0 > \geq -\alpha_0 t\ . \tag{66}$$

First we take $(\rho, m) = (\rho^*, m^*)$. We have

$$\forall(s,t), s \geq 0, t > 0, \alpha_0 s \geq \alpha_0 t\ .$$

We deduce that $\alpha_0$ is non-negative. Assume now that $\alpha_0 = 0$. Then (66) becomes

$$\forall(\rho, m, s),\ < \nabla_{t,x}.(\rho, m), \phi_0 > \geq 0\ . \tag{67}$$

Let $\psi$ be the solution in $L_0^2$ of the following system

$$\begin{cases} \Delta\psi = -\phi_0\ , \\ \partial_t\psi(0,.) = \rho_0\ , \\ \partial_t\psi(T,.) = \rho_T\ . \end{cases}$$

We define $(\rho, m) = \nabla_{t,x}\psi$. We get then $\|\phi_0\|_{L^2}^2 \leq 0$, and therefore $\phi_0 = 0$, which is a contradiction. We deduce that $\alpha_0 > 0$.

Finally, define $\lambda^* = \frac{\phi_0}{\alpha_0}$. We have to check that the triplet $(\rho^*, m^*, \lambda^*)$ is indeed a saddle-point of $L$.

Taking $s = 0$, and letting $t$ goes to 0, we get

$$\forall(\rho, m) \in H(Q; div),\ L(\rho^*, m^*, \lambda^*) \leq L(\rho, m, \lambda^*)\ . \tag{68}$$

Moreover, we have $\nabla_{t,x}(\rho^*, m^*) = 0$, and therefore

$$\forall\lambda \in L_0^2, L(\rho^*, m^*, \lambda) \leq L(\rho^*, m^*, \lambda^*)\ . \tag{69}$$

Hence, the triplet $(\rho^*, m^*, \lambda^*)$ is a saddle-point of $L$, and theorem 2.1 is proved.

# 3 More on the saddle-point

Now we prove some properties of the saddle-point. Indeed, we have

$$K(\rho^*, m^*) = \sup_{(a,b)\in\tilde{K}} \int_{[0;T]} \int_{\mathbb{T}^d} a\rho^* + b.m^* dxdt\ . \tag{70}$$

From the regularity of the minimizer $(\rho^*, m^*)$, and under assumption (53), we get that the optimal pair $(a^*, b^*)$ is actually reached and satisfies in $L^2(Q)$ the following equality

$$\begin{cases} a^* = -\dfrac{|m^*|^2}{2\rho^{*2}} , \\ \\ b^* = \dfrac{m^*}{\rho^*} . \end{cases} \tag{71}$$

Let $(\tilde{\rho}, \tilde{m})$ in $\mathcal{C}^\infty(Q)^{1+d}$ and $\delta$ in $\mathbb{R}$. Assume furthermore that

$$\tilde{\rho}(0, .) = 0,\ \tilde{\rho}(T, .) = 0 . \tag{72}$$

Then the pair $(\rho = \rho^* + \tilde{\rho}, m = m^* + \tilde{m})$ satisfies

$$0 \leq \delta \int_{[0;T]} \int_{\mathbb{T}^d} (a^*\tilde{\rho} + b^*.\tilde{m}) + \lambda^* \nabla_{t,x}.(\tilde{\rho}, \tilde{m}) dxdt + O(\delta^2) . \tag{73}$$

Letting $\delta$ go to 0, and then using the density of $\mathcal{C}^\infty(Q)^{1+d}$ in $H(Q; div)$, we get that for any $(\tilde{\rho}, \tilde{m})$ in $H(Q; div)$ satisfying (72)

$$\int_{\mathbb{T}^d} (a^*\tilde{\rho} + b^*.\tilde{m}) + \lambda^* \nabla_{t,x}.(\tilde{\rho}, \tilde{m}) dxdt = 0 . \tag{74}$$

We remind now that any function $u$ in $L^2(Q)^{1+d}$ can be uniquely (and continuously) decomposed in $L^2(Q)^{1+d}$ as a sum $u = u_1 + u_2$ such that

$$u_1 = \nabla_{t,x}\phi \text{ for some } \phi \in H^1(Q) ,$$

$$u_2 \text{ satisfies } \begin{cases} \nabla_{t,x}.u_2 = 0 & \text{in } L^2(Q) , \\ \\ u_2.n = 0 & \text{on } \partial Q . \end{cases} \tag{75}$$

We write then $(a^*, b^*) = \nabla\phi^* + v^*$. Injecting in (74) and integrating by part, we get that for all $(\tilde{\rho}, \tilde{m})$ in $H(Q; div)$ satisfying (72)

$$\int_{[0;T]} \int_{\mathbb{T}^d} (\lambda^* - \phi^*) \nabla_{t,x}.(\tilde{\rho}, \tilde{m}) dxdt + \int_{[0;T]} \int_{\mathbb{T}^d} v^*.(\tilde{\rho}, \tilde{m}) dxdt = 0 . \tag{76}$$

Hence we get that if $\nabla_{t,x}.(\tilde{\rho}, \tilde{m}) = 0$

$$\int_{[0;T]} \int_{\mathbb{T}^d} v^*.(\tilde{\rho}, \tilde{m}) dxdt = 0 . \tag{77}$$

Integrating by part the product $< v^*, \nabla_{t,x}\phi >$ for any $\phi$ in $H^1(Q)$, and using the definition of $v^*$, we get 0. Then using the decomposition property of $L^2(Q)^{1+d}$ for any $v$ in $L^2(Q)^{1+d}$, we conclude that $v^* = 0$. Then injecting in (76) and using the definition of a saddle-point, we see that $(\rho^*, m^*, \phi^*)$ is a saddle-point of $L$.

As in [1], we are now ready to define a new Lagrangian $\mathcal{L}$. We also take the same notations

$$\begin{cases} \mu = (\rho, m) \\ q = (a, b) \\ F(q) = \begin{cases} 0 \text{ if } q \in \tilde{K} \\ +\infty \text{ else} \end{cases} \\ G(\phi) = \int_{\mathbb{T}^d} [\phi(0,.)\rho_0 - \phi(T,.)\rho_T] \\ < \mu, q >= \int_{[0;T]} \int_{\mathbb{T}^d} \mu.q \; dxdt \; , \end{cases}$$

to get that for all $(\mu, q, \phi) \in L^2(Q)^{d+1} \times L^2(Q)^{d+1} \times H^1(Q)$ (from now on, this space will be denoted by $E(Q)$)

$$\mathcal{L}(\mu, q, \phi) = -F(q) - G(\phi) + < \mu, q - \nabla_{t,x}\phi > \; . \tag{78}$$

We have now the following theorem

**Theorem 3.1.** *$(\mu^*, q^*, \phi^*)$ is a saddle-point of $\mathcal{L}$ in $E(Q)$. Moreover, any saddle-point of $\mathcal{L}$ in $E(Q)$ is of the form $(\tilde{\mu}, q^*, \phi^* + \tilde{C})$, where $\tilde{C}$ is a constant, and $\tilde{\mu}$ is a solution of the time-continuous mass transport problem.*

*Proof.* Let $\mu$ in $L^2(Q)^{1+d}$. We have $< \mu, q^* - \nabla_{t,x}\phi^* >= 0 =< \mu, q^* - \nabla_{t,x}\phi^* >$. Hence we have

$$\mathcal{L}(\mu^*, q^*, \phi^*) \leq \mathcal{L}(\mu, q^*, \phi^*) \; .$$

Let $(q, \phi)$ in $\tilde{K} \times H^1(Q)$. First we observe that a simple integration by part gives $< \mu^*, \nabla_{t,x}\phi >= -G(\phi)$ since $\mu^*$ is in $V(Q)$ and satisfies (7). Then we have

$$\begin{aligned} \mathcal{L}(\mu^*, q, \phi) &= < \mu^*, q - \nabla_{t,x}\phi > -G(\phi) \\ &= < \mu^*, q > \\ &\leq < \mu^*, q^* > \\ &\leq < \mu^*, q^* - \nabla_{t,x}\phi^* > -G(\phi^*) \\ &\leq \mathcal{L}(\mu^*, q^*, \phi^*) \; . \end{aligned}$$

Then summarizing these inequalities, we get that for any $(\mu, q, \phi)$ in $L^2(Q)^{1+d} \times L^2(Q)^{1+d} \times H^1(Q)$, we have

$$\mathcal{L}(\mu^*, q, \phi) \leq \mathcal{L}(\mu^*, q^*, \phi^*) \leq \mathcal{L}(\mu, q^*, \phi^*) \; , \tag{79}$$

which precisely means that $(\mu^*, q^*, \phi^*)$ is a saddle-point of $\mathcal{L}$.

Let $(\tilde{\mu},\tilde{q},\tilde{\phi})$ in $L^2(Q)^{1+d} \times L^2(Q)^{1+d} \times H^1(Q)$ be an other saddle-point of $\mathcal{L}$. We have $\tilde{q} \in \tilde{K}$. Assume next that $\tilde{q} \neq \nabla_{t,x}\tilde{\phi}$, and define $\mu_n = n(\nabla_{t,x}\tilde{\phi} - \tilde{q})$. From the definition of a saddle-point, we get that

$$\begin{aligned}\mathcal{L}(\tilde{\mu},\tilde{q},\tilde{\phi}) &\leq \ < \mu_n, \tilde{q} - \nabla_{t,x}\tilde{\phi} > -G(\tilde{\phi}) \\ &\leq \ -n\|\tilde{q} - \nabla_{t,x}\tilde{\phi}\|^2_{L^2(Q)} - G(\tilde{\phi}) \ .\end{aligned}$$

We obtain a contradiction by letting $n$ go to infinity. Then $\tilde{q}$ is a gradient. Using the decomposition property if $L^2(Q)^{1+d}$, we write $\tilde{\mu} = \mu^* + \bar{\mu} + \nabla_{t,x}\bar{\phi}$. From the definition of a saddle-point, we know that for any $(q,\phi)$ in $L^2(Q)^{1+d} \times H^1(Q)$, we have

$$\mathcal{L}(\tilde{\mu},q,\phi) \leq \mathcal{L}(\tilde{\mu},\tilde{q},\tilde{\phi}) \ . \tag{80}$$

Assume that there exists some $\phi_1$ such that $< \tilde{\mu}, \nabla_{t,x}\phi_1 > +G(\phi_1) < 0$. Then taking $(q,\phi) = (0, n\phi_1)$ in (80), we get

$$-n\,(< \tilde{\mu}, \nabla_{t,x}\phi_1 > +G(\phi_1)) \leq \mathcal{L}(\tilde{\mu},\tilde{q},\tilde{\phi}) \ . \tag{81}$$

Letting $n$ go to infinity, we obtain a contradiction. Hence we see that for all $\phi$ in $H^1(Q)$, $< \tilde{\mu}, \nabla_{t,x}\phi > +G(\phi) = 0$. In particular, this has to be true with $\bar{\phi}$. Using our decomposition of $\tilde{\mu}$, we get

$$\begin{aligned} 0 &= \ < \mu^* + \bar{\mu} + \nabla_{t,x}\bar{\phi}, \bar{\phi} > +G(\bar{\phi}) \\ &= \ < \mu^*, \nabla_{t,x}\bar{\phi} > +G(\bar{\phi}) > + < \bar{\mu}, \nabla_{t,x}\bar{\phi} > + \|\nabla_{t,x}\bar{\phi}\|^2_{L^2(Q)} \ .\end{aligned}$$

Integrating by part, and using the properties of $\mu^*$ and $\bar{\mu}$, we deduce that $\nabla_{t,x}\bar{\phi}$ is null. Hence we see that $\tilde{\mu}$ is in fact in $V(Q)$ and satisfies the boundary conditions (7). We have now to prove that $\tilde{q} = q^*$. From the fact that both pairs have to define saddle-points, we get

$$< \tilde{\mu}, q^* - \tilde{q} > \ = \ 0 \ , \tag{82}$$
$$< \mu^*, q^* - \tilde{q} > \ = \ 0 \ , \tag{83}$$
$$< \mu^*, q^* > \ = \ < \tilde{\mu}, \tilde{q} > \ . \tag{84}$$

The identity $\tilde{q} = q^*$ follows quite obviously from (83). Indeed, the set $\tilde{K}$ is strictly convex, and the $L^\infty$ bounds on $q^*$ ensure that we have some kind of uniform strict convexity. Precisely, we have

**Lemma 3.2.** *Let $\tilde{q}$ in $L^2(Q)$ such that $< \mu^*, q^* - \tilde{q} >= 0$. Then $\tilde{q} = q^*$ a.e. on $Q$.*

*Proof.* Since $q^*$ is bounded in $L^\infty(Q)$, we can have uniform estimates on the strict convexity of $\tilde{K}$. The geometric intuition is easily understood when looking at Figure 1.
Let $\epsilon > 0$. There exists some $\delta > 0$ such that for any $(t,x)$ in $Q$

$$|q^* - \tilde{q}| > \epsilon \Rightarrow \mu^*.(q^* - \tilde{q}) < -\delta\|\mu^*\| \ . \tag{85}$$

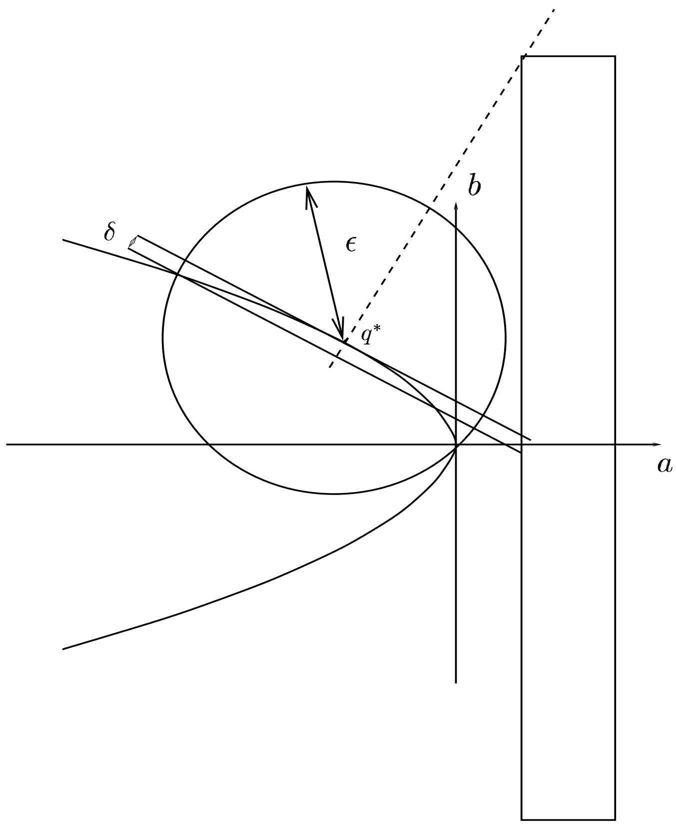


Figure 1: Geometrical intuition

Let $A_\epsilon = \{(t,x) \in Q \text{ s.t. } |q^* - \tilde{q}| > \epsilon\}$. We have

$$\begin{aligned} 0 &= \ < \mu^*, q^* - q > \ , \\ &\leq \int_{A_\epsilon} \mu^*.(q^* - q) dt dx \ , \\ &\leq -\delta \int_{A_\epsilon} \|\mu^*\| \ , \\ &\leq -\delta\alpha_1 |A_\epsilon| \ . \end{aligned}$$

Then we get that $|A_\epsilon| = 0$. This achieves the proof of the lemma. □

Finally, since $(\tilde{\mu}, \tilde{q}, \tilde{\phi})$ is a saddle-point of $\mathcal{L}$, we have

$$< \tilde{\mu}, \tilde{q} >= K(\tilde{\rho}, \tilde{m}) \ . \tag{86}$$

Then we get from (84) (and from the fact that $\tilde{\mu}$ is in $V(Q)$ and satisfies (7)) that $\tilde{\mu}$ is a solution of the time-continuous mass transport problem. This achieves the proof of theorem 3.1. □

This theorem answers a question that was left open in [1] on the existence of the saddle-point for $\mathcal{L}$ in the infinite dimensional case. Moreover, we give a precise functional background to search this saddle-point.

**Remark 3.3.** *In theorem 3.1, the variable $\mu$ is taken in $L^2(Q)^{1+d}$ rather than in $H(Q;div)$. This will remove a constraint when the question will turn to the effective search of the saddle-point.*

# 4 On the algorithm of [1]

In [1], the authors defined an augmented Lagrangian as a preliminary for their numerical method. To get some more coercivity they perturbed the functional $F$. Here, since we already made an assumption of boundedness away from zero on the saddle-point, it is not necessary. We define then on $E(Q)$ the augmented Lagrangian

$$\begin{aligned} L_r(\mu,q,\phi) &= F(q)+G(\phi)+<\mu,\nabla_{t,x}\phi-q> \\ &\quad +\frac{r}{2}<\nabla_{t,x}\phi-q,\nabla_{t,x}\phi-q>\ . \end{aligned} \tag{87}$$

where $r$ is a positive parameter.

**Remark 4.1.** *We can change the signs since we proved the existence of a saddle-point of $\mathcal{L}$. In this formulation, the constraint is to get $q$ as a gradient. This constraint has been augmented, instead of the old constraint that $\mu$ is in $V(Q)$ and satisfies (7).*

It is a classical result that if $(\mu,q,\phi)$ is a saddle-point of $L_r$, it is also a saddle-point of $\mathcal{L}$ (and conversely). J.D. Benamou and Y. Brenier used then a numerical algorithm **ALG2** to solve the problem. We remind here this algorithm and refer to [1] for more explanations on the steps and some numerical results. Here we are now concerned with the convergence of the method in the continuous case.

**ALG2:**

- $(\phi^{n-1},q^{n-1},\mu^n)$ are given.
- Step A: Find $\phi^n$ in $H^1(Q)\cap L^2_0(Q)$ such that :

$$L_r(\phi^n,q^{n-1},\mu^n)\le L_r(\phi,q^{n-1},\mu^n),\ \ \forall\phi. \tag{88}$$

- Step B: Find $q^n$ in $L^2(Q)^{1+d}$ such that :

$$L_r(\phi^n,q^n,\mu^n)\le L_r(\phi^n,q,\mu^n),\ \ \forall q. \tag{89}$$

- Step C : Do

$$\mu^{n+1} = \mu^n + \delta(\nabla_{t,x}\phi^n - q^n) \tag{90}$$

(where $r > 0$ is the parameter of the Augmented Lagrangian).

- Go back to step A.

**Remark 4.2.** *In step A, the minimization is performed over $H^1(Q) \cap L^2_0(Q)$ in order to have a unique solution. Moreover, it is a way to fix the additive constant from theorem 3.1.*

The convergence of the sequence constructed by ALG2 is proved in [7] under some quite general assumptions, which are unfortunately not fully satisfied here. However, there proof can be adapted in order to deal with the problem under consideration.

**Theorem 4.3.** *Assume that*

$$0 < \delta < \delta_M = \frac{1+\sqrt{5}}{2} r \; . \tag{91}$$

*Then the sequence constructed by ALG2 satisfies the following convergence properties*

$$\phi^n \to \phi^* \text{ strongly in } H^1(Q) \; , \tag{92}$$

$$q^n \to q^* \text{ strongly in } L^2(Q)^{1+d} \; , \tag{93}$$

$$\mu^{n+1} - \mu^n \to 0 \text{ strongly in } L^2(Q)^{1+d} \; , \tag{94}$$

$$\mu^n \text{ is bounded in } L^2(Q)^{1+d} \; . \tag{95}$$

*Moreover, if $\tilde{\mu}$ is a (weak) cluster point of $(\mu^n)$ in $L^2(Q)^{1+d}$, then $(\tilde{\mu}, q^*, \phi^*)$ is a saddle-point of $L_r$ on $E(Q)$.*

*Proof.* The proof of the convergence of ALG2 in [7] require some uniform convexity properties for $F$, which are not satisfied here. However, due to the particular form of our function $F$, the first part of their proof simplifies (since any term dealing with $F$ turns to be null). Hence a simple rewriting of their (intricate) calculations leads to ($|x|$ denote the $L^2$ norm of $x$)

$$\left\{ \begin{array}{l} (|\bar{\mu}^n|^2 + \delta r|\bar{q}^{n-1}|^2) - (|\bar{\mu}^{n+1}|^2 + \delta r|\bar{q}^n|^2) \geq \delta(2r-\delta)|\nabla_{t,x}\bar{\phi}^n - \bar{q}^n|^2 \\ \\ +\delta r|\bar{q}^n - \bar{q}^{n-1}|^2 - \delta|r-\delta|(\dfrac{1}{\alpha}|\nabla_{t,x}\bar{\phi}^{n-1} - \bar{q}^{n-1}|^2 + \alpha|\bar{q}^n - \bar{q}^{n-1}|^2) \; , \end{array} \right. \tag{96}$$

where $\bar{\mu}^n = \mu^n - \mu^*$, $\bar{q}^n = q^n - q^*$, $\bar{\phi}^n = \phi^n - \phi^*$ and $\alpha > 0$ is a parameter.

If $0 < \delta \le r$, taking $\alpha = 1$ and observing that $|r-\delta| = r-\delta$, we get

$$v_{n-1} - v_n \ge \delta r |\nabla_{t,x}\bar{\phi}^n - \bar{q}^n|^2 + \delta^2 |\bar{q}^n - \bar{q}^{n-1}|^2 \; , \tag{97}$$

with $v_n = (|\bar{\mu}^{n+1}|^2 + \delta r |\bar{q}^n|^2 + \delta(r-\delta)|\nabla_{t,x}\bar{\phi}^n - \bar{q}^n|^2)$.

If $r < \delta < \delta_M$, taking $\alpha = \frac{1+\sqrt{5}}{2}$, we have

$$w_{n-1} - w_n \ge \frac{\delta_M \delta}{r}(\delta_M - \delta)|\nabla_{t,x}\bar{\phi}^n - \bar{q}^n|^2 + \delta(\delta_M - \delta)|\bar{q}^n - \bar{q}^{n-1}|^2 \; , \tag{98}$$

with $w_n = (|\bar{\mu}^{n+1}|^2 + \delta r |\bar{q}^n|^2 + \dfrac{\delta r}{\delta_M}(\delta - r)|\nabla_{t,x}\bar{\phi}^n - \bar{q}^n|^2)$.

In (97) and (98), the right-hand sides are non-negative. Then the sequences $(v_n)$ and $(w_n)$ are decreasing. Hence we have that $(\mu_n)$ and $(q_n)$ are uniformly bounded in $L^2(Q)^{1+d}$. Moreover, we see from the right-hand sides that the series $\sum_{n=1}^{\infty} |\nabla_{t,x}\bar{\phi}^n - \bar{q}^n|^2$ and $\sum_{n=1}^{\infty} |\bar{q}^n - \bar{q}^{n-1}|^2$ are finite. It implies that

$$\begin{cases} \nabla_{t,x}\bar{\phi}^n - \bar{q}^n \to 0 \text{ strongly in } L^2(Q)^{1+d} \; , \\ (\bar{q}^n) \text{ is a cauchy sequence in } L^2(Q)^{1+d} \; . \end{cases} \tag{99}$$

We have then that $(q_n)$ converges strongly in $L^2(Q)^{1+d}$ to some $\tilde{q}$. Hence we have that $(\nabla_{t,x}\phi_n)$ converges also strongly to $\tilde{q}$ in $L^2(Q)^{1+d}$. Since $\phi_n$ is in $L^2_0(Q)$, we see then that the sequence $(\phi_n)$ converges strongly in $H^1(Q) \cap L^2_0(Q)$ to some $\tilde{\phi}$. Moreover, the sequence $(\mu_n)$ is bounded in $L^2(Q)^{1+d}$. We can then extract a subsequence (still denoted by $n$) weakly converging in $L^2(Q)^{1+d}$ to some $\tilde{\mu}$. We have now to prove that $\tilde{q} = q^*$ and $\tilde{\phi} = \phi^*$.

Step A means that for any $\phi$ in $H^1(Q)$

$$\begin{aligned} 0 \;\; \le \;\; & G(\phi) - G(\phi^n) + < \mu^n, \nabla_{t,x}\phi - \nabla_{t,x}\phi^n > + \\ & r < \nabla_{t,x}\phi^n - q^{n-1}, \nabla_{t,x}\phi - \nabla_{t,x}\phi^n > \; . \end{aligned} \tag{100}$$

Step B means that for any $q$ in $L^2(Q)^{1+d}$

$$0 \;\; \le \;\; F(q) - < \mu^n, q - q^n > + r < q^n - \nabla_{t,x}\phi^n, q - q^n > \; . \tag{101}$$

Taking $\phi = \phi^*$ in (100) and $q = q^*$ in (101), and letting $n$ go to infinity, we get

$$0 \;\; \le \;\; G(\phi^*) - G(\tilde{\phi}) + < \tilde{\mu}, \nabla_{t,x}\phi^* - \nabla_{t,x}\tilde{\phi^n} > , \tag{102}$$

$$0 \;\; \le \;\; - < \tilde{\mu}, q^* - \tilde{q} > \; . \tag{103}$$

Then adding the two inequalities, and using that $\nabla_{t,x}\tilde{\phi} = \tilde{q}$ and $\nabla_{t,x}\phi^* = q^*$, we get

$$G(\tilde{\phi}) \leq G(\phi^*) \ . \tag{104}$$

Moreover, since $(\mu^*, q^*, \phi^*)$ is a saddle-point of $L_r$, we have

$$G(\phi^*) \leq G(\tilde{\phi}) \ . \tag{105}$$

We deduce that $G(\phi^*) = G(\tilde{\phi})$. We remind now that for any $\phi$ in $H^1(Q)$, $G(\phi)+ < \mu^*, \nabla_{t,x}\phi >= 0$. Using this equality with $\phi^*$ and $\tilde{\phi}$, we get

$$< \mu^*, q^* >=< \mu^*, \tilde{q} > \ . \tag{106}$$

We remind now lemma 3.2 to get $\tilde{q} = q^*$. Hence $\tilde{\phi} = \phi^*$. To end the proof of theorem 4.3, it remains to show that $(\tilde{\mu}, q^*, \phi^*)$ is a saddle-point of $L_r$. Letting $n$ go to infinity in (100) and (101) and adding the resulting inequalities, we get that for any $(q, \phi)$ in $L^2(Q)^{1+d} \times H^1(Q)$,

$$G(\phi^*) \leq F(q) + G(\phi)+ < \tilde{\mu}, \nabla_{t,x}\phi - q > \ . \tag{107}$$

We conclude that $(\tilde{\mu}, q^*, \phi^*)$ is a saddle-point of $\mathcal{L}$ and then also a saddle-point of $L_r$. This achieves the proof of theorem 4.3. □

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